

# RATIONALITY OF MOTIVIC ZETA FUNCTION AND CUT-AND-PASTE PROBLEM

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ABSTRACT. Assuming the positive solution to the Cut-and-paste problem we prove that the motivic zeta function remains irrational after inverting  $\mathbb{L}$ .

## 1. INTRODUCTION

Fix a field  $\mathbb{F}$  and let  $K_0[\mathcal{V}_{\mathbb{F}}]$  denote the Grothendieck ring of varieties over  $\mathbb{F}$ . That is  $K_0[\mathcal{V}_{\mathbb{F}}]$  is the abelian group which is generated by isomorphism classes of  $\mathbb{F}$ -varieties with relations

$$[X] = [Y] + [X \setminus Y]$$

if  $Y \subset X$  is a closed subvariety. The product in  $K_0[\mathcal{V}_{\mathbb{F}}]$  is defined as

$$[X] \cdot [Y] = [X \times_{\mathbb{F}} Y]$$

In [LaLu1] we have asked the following question:

*Cut-and-paste problem.* Let  $Z_1, \dots, Z_k; W_1, \dots, W_l$  be  $\mathbb{F}$ -varieties and consider the disjoint unions  $X = \coprod Z_i$  and  $Y = \coprod W_j$ . Suppose that  $[X] = [Y]$ . Is it possible to decompose  $X$  and  $Y$  into locally closed subvarieties

$$X = \coprod_{i=1}^k X_i, \quad Y = \coprod_{i=1}^k Y_i$$

such that for each  $i$  the varieties  $X_i$  and  $Y_i$  are isomorphic?

Some positive results for this problem are obtained in the paper [LiSeb]. They prove that the solution to the problem is positive (in characteristic zero) if 1)  $\dim X \leq 1$ , 2)  $X$  is a smooth connected projective surface, 3)  $X$  contains only finite many rational curves.

In this note we want to relate the Cut-and-paste problem to the question of rationality of the motivic zeta function

$$\zeta_X(t) = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n \in K_0[\mathcal{V}_{\mathbb{F}}][[t]]$$

This motivic zeta function was introduced by Kapranov in [Ka], where he proves that  $\zeta_X(t)$  is rational if  $\dim X \leq 1$ . He also says that it is natural to expect rationality of  $\zeta_X(t)$  for any variety  $X$ .

This conjecture of Kapranov was disproved in [LaLu1] and [LaLu2], where we show that the motivic zeta function of a surface  $X$  is rational if and only if  $X$  has Kodaira dimension  $-\infty$  (for  $\mathbb{F} = \mathbb{C}$ ). The proof of this uses a ring homomorphism  $K_0(\mathcal{V}_{\mathbb{C}}) \rightarrow \mathcal{H}$  to a field  $\mathcal{H}$  which factors through the quotient  $K_0(\mathcal{V}_{\mathbb{C}})/\mathbb{L}$ , where  $\mathbb{L} = [\mathbb{A}^1]$ . Hence the question of rationality of the motivic zeta function in the localized ring  $K_0[\mathcal{V}_{\mathbb{F}}][\mathbb{L}^{-1}]$  is still open.

In the paper [DeLoe] the authors conjecture (Conjecture 7.5.1) that  $\zeta_X(t)$  is rational in  $K_0[\mathcal{V}_{\mathbb{F}}][\mathbb{L}^{-1}]$ .

In this article we prove that the positive solution to the Cut-and-paste problem implies that  $\zeta_X(t)$  is *not* rational in  $K_0[\mathcal{V}_{\mathbb{F}}][\mathbb{L}^{-1}]$ . This follows easily from our results in [LaLu1].

We thank Ravi Vakil, whose beautiful recent lecture in Indiana University on motivic Grothendieck ring prompted us to think again about the subject.

## 2. RATIONALITY OF POWER SERIES WITH COEFFICIENTS IN A RING

Let  $A$  be a commutative ring with 1. We recall and compare various notions of rationality of power series with coefficients in  $A$ .

**Definition 2.1.** A power series  $f(t) \in A[[t]]$  is **globally rational** if and only if there exist polynomials  $g(t), h(t) \in A[t]$  such that  $f(t)$  is the unique solution of  $g(t)x = h(t)$ .

**Definition 2.2.** A power series  $f(t) = \sum_{i=0}^{\infty} a_i t^i \in A[[t]]$  is **determinantly rational** if and only if there exist integers  $m$  and  $n$  such that

$$\det \begin{pmatrix} a_i & a_{i+1} & \cdots & a_{i+m} \\ a_{i+1} & a_{i+2} & \cdots & a_{i+m+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i+m} & a_{i+m+1} & \cdots & a_{i+2m} \end{pmatrix} = 0$$

for all  $i > n$ .

It is classical that the Definition 2.1 is equivalent to Definition 2.2 if  $A$  is a field.

**Definition 2.3.** A power series  $f(t) \in A[[t]]$  is **pointwise rational** if and only if for all homomorphisms  $\Phi$  from  $A$  to a field,  $\Phi(f)$  is rational by either of the two previous definitions.

These definitions are related by the following proposition [LaLu2], Prop. 2.4:

**Proposition 2.4.** Any globally rational power series is determinantly rational, and any determinantly rational power series is pointwise rational. Neither converse holds for a general coefficient ring  $A$ . All three conditions are equivalent when  $A$  is an integral domain.

It is known that the ring  $K_0[\mathcal{V}_{\mathbb{F}}]$  has zero divisors [Po].

3. CUT-AND-PASTE PROBLEM AND RATIONALITY OF  $\zeta_X(t)$ 

The following theorem was proved in [LaLu2], Thm. 7.6 and Cor. 3.8:

**Theorem 3.1.** *Let  $X$  be a complex surface of Kodaira dimension  $\geq 0$ . Then the zeta function  $\zeta_X(t) \in K_0[\mathcal{V}_{\mathbb{C}}][[t]]$  is not pointwise rational.*

On the positive side it is relatively easy to prove the following theorem [LaLu2], Thm. 3.9:

**Theorem 3.2.** *If  $X$  is a surface with the Kodaira dimension  $-\infty$ , then the zeta function  $\zeta_X(t) \in K_0[\mathcal{V}_{\mathbb{C}}][[t]]$  is globally rational.*

Let  $Y$  be a smooth projective variety of dimension  $d$ . Recall that the polynomial

$$h_Y(s) := 1 + h^{1,0}(Y)s + h^{2,0}(Y)t^2 + \dots + h^{d,0}(Y)s^d$$

is a birational invariant of  $Y$  [Hart], Ch. II, Exercise 8.8. Here  $h^{i,0}(Y) = \dim H^0(Y, \Omega_Y^i)$ . Therefore we may (in characteristic zero) define  $h_Z(t)$  for any variety  $Z$ , not necessarily smooth and projective, as

$$h_Z(s) = h_Y(s)$$

where  $Y$  is any smooth projective model of  $Z$ . The Künneth formula for the Hodge structure on the cohomology of the constant sheaf  $\mathbb{C}$  implies that  $h_Y(s)$  is even a stable birational invariant of  $Y$ , i.e.

$$h_Y(s) = h_{Y \times \mathbb{P}^n}(s)$$

The integer  $P_g(Y) := h^{d,0}(Y)$  is the *geometric genus* of  $Y$ .

Here we prove the following theorem:

**Theorem 3.3.** *Let  $X$  be a complex surface with  $P_g(X) \geq 2$ . Assume that the Cut-and-paste problem has a positive solution. Then the zeta function  $\zeta_X(t) \in K_0[\mathcal{V}_{\mathbb{C}}][\mathbb{L}^{-1}][[t]]$  is not determinantly rational.*

*Proof.* Put  $X^{(n)} := \text{Sym}^n X$ . If the zeta function  $\zeta_X(t) \in K_0[\mathcal{V}_{\mathbb{C}}][\mathbb{L}^{-1}][[t]]$  is determinantly rational then there exist integers  $n > 0$  and  $n_0 > 0$  such that for each  $m > n_0$  the determinant

$$(3.1) \quad \det \begin{pmatrix} X^{(m)} & X^{(m+1)} & \dots & X^{(m+n)} \\ X^{(m+1)} & X^{(m+2)} & \dots & X^{(m+n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ X^{(m+n)} & X^{(m+n+1)} & \dots & X^{(m+2n)} \end{pmatrix}$$

equals zero in the ring  $K_0[\mathcal{V}_{\mathbb{C}}][\mathbb{L}^{-1}]$ . This determinant is the sum

$$(3.2) \quad \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) X^{(m-1+\sigma(1))} \times X^{(m+\sigma(2))} \times \dots \times X^{(m+n-1+\sigma(n+1))}$$

The assumption that the determinant is zero in  $K_0[\mathcal{V}_{\mathbb{C}}][\mathbb{L}^{-1}]$  means that the quantity 3.2 when multiplied by some power  $\mathbb{L}^N$  is zero in  $K_0[\mathcal{V}_{\mathbb{C}}]$ . Then the positive solution to the Cut-and-paste problem implies that the various products in the alternating sum 3.2 when multiplied by  $\mathbb{L}^N$  become pairwise birational (since all of them have the same dimension). Note that the product

$$X^{(m)} \times X^{(m+2)} \times \dots \times X^{(m+2n)}$$

appears exactly once in 3.2. Now we get a contradiction with the following claim, which is proved on p. 11 in [LaLu1]:

*Claim.* For infinitely many  $m > 0$  the equality

$$P_g(X^{(m)} \times \dots \times X^{(m+2n)}) = P_g(X^{(m-1+\sigma(1))} \times \dots \times X^{(m+n-1+\sigma(n+1))})$$

implies that  $\sigma = 1$ . □

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